Integrals of Recursive Functions

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Introduction

In this paper, I would like to present a method that can be used to solve integrals of recursive functions, hopefully in a general sense. I will start by describing the method and its steps, and then I will proceed to an example.

The Method Itself

The fundamental idea behind this method is largely converting the input variable (usually denoted by x) inside the integrand into a quantity that is in terms of the function variable (in this paper, I will denote it g). The purpose of this step is mainly to substitute a variation of differential dg in for the differential dx, as will become apparent in the example. The next step is to derive several expressions that relate g to its derivative(s) that will allow for a much simpler indefinite integral to compute. After this step, we are left with a series of much simpler integrals to compute, which subsequently leaves us with the answer.

Example

$$\int \sqrt{x + \sqrt{x + \sqrt{x + ...}}}\, dx$$

Part 1: Rewrite the Integrand in Recursive Form

In our integrand, we have an infinitely nested chain of $\sqrt{x}$ sums, one inside of the other. We can rewrite this recursively, allowing g to represent the recursive function, and x to represent the input variable.

$$g = \sqrt{x+g}$$

Part 2: Derive Several Substitutable Relations

This is probably the most rigorous, but fairly simple, step of solving the problem. Realize that our original integral can be written in the form of our recursive function.

$$\int \sqrt{x + \sqrt{x + \sqrt{x + ...}}}\, dx = \int g\ dx = \int \sqrt{x+g}\ dx$$

The most critical part of this step is to rewrite g, the integrand of $\int g\ dx$, into a form that contains only one type of variable. It might seem that we are already finished since the current integrand is already in terms of g. However, it is not quite obvious how we should proceed further to solve $\int \sqrt{x+g}\ dx$. Now, we must rewrite $\sqrt{x+g}$ as a quantity that is in terms of g.

$$g = \sqrt{x+g} \tag{1}$$

$$x = g^2 - g \tag{2}$$

Now, we will perform a series of manipulations on equation (1) in order to obtain an equation where g is equal to something in terms of g. First, take the derivative of of g.

$$g' = \frac{1}{2\sqrt{x+g}} \cdot (1 + g') = \frac{1+g'}{2g}$$

Next, make the following deductions.

$$g' = \frac{1+g'}{2g}$$

$$2g = \frac{1+g'}{g'} = \frac{1}{g'} + 1$$

$$2g - 1 = \frac{1}{g'} \tag{3}$$

$$2g = \frac{1}{g'} + 1$$

$$g = \frac{1}{2}\left(\frac{1}{g'} + 1\right) \tag{4}$$

Now, we have solved for g, whose value we can substitute in for g into the integral $\int g\ dx$. Next, we need to solve for dx in terms of g. Let begin with equation (2) and take its derivative.

$$x = g^2 - g$$

$$dx = 2gg' - g' = g' \cdot (2g - 1)$$

$$dx = (2g - 1)\ dg \tag{5}$$

Now, we have solved for the dx component of the integral $\int g\ dx$. Let us now substitute the values of g and dx into the integral, using equations (4) and (5), respectively.

$$\begin{aligned} \int g\ dx &= \int \frac{1}{2}\left(\frac{1}{g'} + 1\right)(2g - 1)\ dg \\ &= \frac{1}{2}\int \left(\frac{g' + 1}{g'}\right)(2g - 1)\ dg \end{aligned}$$

Recall equation (3), which will allow us to write $2g - 1$ in for the $\frac{1}{g'}$ inside the current integrand, leaving us with the following.

$$\begin{aligned} \frac{1}{2}\int \left(\frac{g' + 1}{g'}\right)(2g - 1)\ dg &= \frac{1}{2}\int (g' + 1)(2g - 1)^2\ dg \\ &= \frac{1}{2}\int (g' + 1)\left(4g^2 - 4g + 1\right)\ dg \\ &= \frac{1}{2}\int \left(4g^2g' - 4gg' + g' + 4g^2 - 4g + 1\right)\ dg \end{aligned}$$

Part 3: Split the Integral and Solve the Individuals

Now, let us split what we have so far into 6 separate integrals.

$$\frac{1}{2}\int \left(4g^2g' - 4gg' + g' + 4g^2 - 4g + 1\right)\ dg$$

$$= \frac{1}{2}\left(\int 4g^2g'\ dg - \int 4gg'\ dg + \int g'\ dg + \int 4g^2\ dg - \int 4g\ dg + \int dg\right)$$

Solve Integral 1: $\int 4g^2g'\ dg$

Begin by rewriting g' as $\frac{1}{2g-1}$, from equation (3). We now have the following.

$$4\int \frac{g^2}{2g-1}\ dg$$

Now, perform the following u-substitution.

$$u = 2g-1, \quad g^2 = \left(\frac{u+1}{2}\right)^2$$

$$\frac{du}{dg} = 2$$

$$dg = \frac{du}{2}$$

$$\begin{aligned}
4\int \frac{g^2}{2g-1}\ dg &= 2\int \frac{\left(\frac{u+1}{2}\right)^2}{u}\ du \\
&= 2\int \frac{(u+1)^2}{4u}\ du \\
&= 2\int \frac{u^2+2u+1}{4u}\ du \\
&= 2\int \left(\frac{u^2}{4u} + \frac{1}{2} + \frac{1}{4u}\right)\ du \\
&= 2\left(\frac{u^2}{8} + \frac{u}{2} + \frac{1}{4}\ln|4u|\right) + A \\
&= \frac{(2g-1)^2}{4} + 2g - 1 + \frac{1}{2}\ln|8g-4| + A
\end{aligned}$$

Solve Integral 2: $\int 4gg'\ dg$

Similar to Integral 1, begin by rewriting g' as $\frac{1}{2g-1}$, from equation (3). We now have the following.

$$4\int \frac{g}{2g-1}\ dg$$

Perform the following u-substitution.

$$u = 2g - 1, \quad g = \frac{u+1}{2}$$

$$\frac{du}{dg} = 2$$

$$dg = \frac{du}{2}$$

$$\begin{aligned}
4 \int \frac{g}{2g-1}\, dg &= 2 \int \frac{\left(\frac{u+1}{2}\right)}{u}\, du \\
&= 2 \int \frac{u+1}{2u}\, du \\
&= \int \frac{u+1}{u}\, du \\
&= \int \left(1 + \frac{1}{u}\right)\, du \\
&= u + \ln|u| + B \\
&= 2g - 1 + \ln|2g-1| + B
\end{aligned}$$

Solve Integral 3: $\int g'\, dg$

This can be rewritten as $\int \frac{1}{2g-1}\, dg$, based on equation (3). After completing a u-substitution, allowing $u = 2g - 1$, we have the result $\frac{1}{2} \ln|2g-1| + C$.

Solve Integral 4: $\int 4g^2\, dg$

After using the power rule for integrals, we have $\frac{4}{3}g^3 + D$.

Solve Integral 5: $\int 4g\, dg$

After using the power rule for integrals, we have $2g^2 + E$.

Solve Integral 6: $\int dg$

This integral is quite simple, leaving us with $g + F$.

Part 4: Wrapping Up

All that is left is to combine the individual integrals and write the final answer.

$$\int \sqrt{x+\sqrt{x+\sqrt{x+...}}}\,dx = \frac{1}{2}\int \left(4g^2g' - 4gg' + g' + 4g^2 - 4g + 1\right)\,dg$$

$$= \tfrac{1}{2}\left[\tfrac{(2g-1)^2}{4} + 2g - 1 + \tfrac{1}{2}\ln|8g-4| - (2g - 1 + \ln|2g-1|) + \tfrac{1}{2}\ln|2g-1| + \tfrac{4}{3}g^3 - 2g^2 + g\right] + H$$

$$= \tfrac{1}{2}\left[\tfrac{(2g-1)^2}{4} + 2g - 1 + \tfrac{1}{2}\ln|8g-4| - 2g + 1 - \ln|2g-1| + \tfrac{1}{2}\ln|2g-1| + \tfrac{4}{3}g^3 - 2g^2 + g\right] + H$$

$$= \tfrac{1}{2}\left[\tfrac{4}{3}g^3 + \tfrac{(2g-1)^2}{4} - 2g^2 + g + \tfrac{1}{2}\ln|4(2g-1)| - \ln|2g-1| + \tfrac{1}{2}\ln|2g-1|\right] + H$$

$$= \tfrac{1}{2}\left[\tfrac{4}{3}g^3 + \tfrac{(2g-1)^2}{4} - 2g^2 + g + \tfrac{1}{2}\ln|4| + \tfrac{1}{2}\ln|(2g-1)| - \ln|2g-1| + \tfrac{1}{2}\ln|2g-1|\right] + H$$

$$= \tfrac{1}{2}\left[\tfrac{4}{3}g^3 + \tfrac{(2g-1)^2}{4} - 2g^2 + g + \tfrac{1}{2}\ln|4|\right] + H$$

$$= \tfrac{2}{3}g^3 + \tfrac{(2g-1)^2}{8} - g^2 + \tfrac{1}{2}g + \tfrac{1}{4}\ln|4| + H$$

$$= \tfrac{2}{3}g^3 + \tfrac{(2g-1)^2}{8} - g^2 + \tfrac{1}{2}g + H$$

$$= \tfrac{2}{3}g^3 + \left(\tfrac{g^2}{2} - \tfrac{g}{2} + \tfrac{1}{8}\right) - g^2 + \tfrac{1}{2}g + H$$

$$\boxed{= \frac{2}{3}g^3 - \frac{1}{2}g^2 + H}$$

Our result is a polynomial, where the input variable is g, completely eliminating the concept of a recursive component.

Part 5: A Note on the Limits of Integration

One point to note is that the limits of integration must be converted from x-limits to g-limits, if evaluating the definite integral. However, a problem seems to arise when converting the limits of integration for this integral. For example, let the x-limits be 0 and 1, leaving us with the following.

$$\int_0^1 \sqrt{x+\sqrt{x+\sqrt{x+...}}}\,dx$$

To convert to g-limits, we can use equation (2) to solve for the g-limits. Set $g^2 - g = 0$ and $g^2 - g = 1$ and solve. Solving the first quadratic equation yields solutions $g = 0$ and $g = 1$. Solving the second equation yields solutions $g = \frac{1 \pm \sqrt{5}}{2}$. Hence, we have 4 combinations of g-limits that we could possibly input. For the first quadratic equation, we can eliminate the solution $g = 0$, since that would result in an undefined quantity in the equation located 2 steps before equation (3). For the second quadratic, g could never be negative according to the definition made in equation (1), because we restrict g to the domain $g \in [0, \infty)$. Therefore, we can eliminate the limit $\frac{1-\sqrt{5}}{2}$. Thus, the final g-limits are $g = 1$ and $\frac{1+\sqrt{5}}{2}$.