Proofs - 3

Vignesh Nydhruva

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Question

The following question is obtained from the textbook: *Discrete Mathematics and its Applications* 8th ed. by Kenneth H. Rosen.

> Prove that
> $$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n} \quad \forall\{n\in\mathbb{Z}|n\geq 0\}$$

Step 1: Basis

We will attempt a proof by mathematical induction. First, we must prove the base case for $n = 0$, since that establishes the validity of the proof from the most simple case. $P(n)$ represents the propositional function for the proposed equation for all cases. $P(n)$ must either be true or false. If we show that $P(n)$ outputs true for all n, then we have proved the statement to be true for all n. $P(0)$ is represented by the following equation:

$$\sum_{j=0}^{0}\left(-\frac{1}{2}\right)^{j} = \frac{2^{0+1}+(-1)^0}{3\cdot 2^0}$$

The left-hand side simplifies to $\left(-\frac{1}{2}\right)^0$. Now, we must show that the right-hand side of the equation is also equal to $\left(-\frac{1}{2}\right)^0$.

$$\frac{2^{0+1}+(-1)^0}{3\cdot 2^0} = \frac{2+1}{3\cdot 2^0}$$

$$= \frac{3}{3 \cdot 1}$$
$$= 1$$
$$= \left(-\frac{1}{2}\right)^0$$

Now, we have proved the base case to be true.

Step 2: Inductive Step

We must now prove the general case. First, assume $P(k)$ to be true for any arbitrary k.

$$P(k) : \sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j = \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k}$$

Now, we need to prove that $P(k+1)$ is true. Doing this will prove the statement for all n. This is because, k is an arbitrary value that shares the domain of n. So, if we have proved the statement for any arbitrary value, we can connect that arbitrary value to any definite value of n and the statement will hold. Therefore, we will have proved the equation. Now, perform the following simplifications.

$$P(k+1) : \sum_{j=0}^{k+1} \left(-\frac{1}{2}\right)^j = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}$$
$$= \sum_{j=0}^{k} \left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{k+1}$$
$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \left(-\frac{1}{2}\right)^{k+1}$$
$$= \frac{2^{k+1} + (-1)^k}{3 \cdot 2^k} + \left(-\frac{1}{2}\right)^{k+1} \cdot \frac{3 \cdot 2^k}{3 \cdot 2^k}$$
$$= \frac{2^{k+1} + (-1)^k + 3 \cdot 2^k \cdot \left(-\frac{1}{2}\right)^{k+1}}{3 \cdot 2^k}$$
$$= \frac{2^{k+1} + (-1)^k + 3 \cdot 2^k \cdot (-1)^{k+1} \cdot \left(\frac{1}{2}\right)^{k+1}}{3 \cdot 2^k}$$
$$= \frac{2^{k+1} + (-1)^k + \frac{3}{2} \cdot (-1)^{k+1}}{3 \cdot 2^k}$$

$$
\begin{aligned}
&= \frac{\frac{2}{2} \cdot 2^{k+1} + \frac{2}{2} \cdot (-1)^k + \frac{3}{2} \cdot (-1)^{k+1}}{3 \cdot 2^k} \\
&= \frac{1}{3 \cdot 2^k} \cdot \frac{2^{k+2} + 2 \cdot (-1)^k + 3 \cdot (-1)^{k+1}}{2} \\
&= \frac{2^{k+2} + 2 \cdot (-1)^k + 3 \cdot (-1)^{k+1}}{3 \cdot 2^{k+1}} \\
&= \frac{2^{k+2} + \frac{(-1)}{(-1)} \cdot 2 \cdot (-1)^k + 3 \cdot (-1)^{k+1}}{3 \cdot 2^{k+1}} \\
&= \frac{2^{k+2} - 2 \cdot (-1)^{k+1} + 3 \cdot (-1)^{k+1}}{3 \cdot 2^{k+1}} \\
&= \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}
\end{aligned}
$$

Notice that we have now shown that the proposition $P(k+1)$ holds with the assumption that $P(k)$ is true. Therefore, we have proven that

$$
\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n} \quad \forall \{n \in \mathbb{Z} | n \geq 0\} \quad \square.
$$