Proofs - 2

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Question

The following question is obtained from the textbook: *Discrete Mathematics and its Applications* 8th ed. by Kenneth H. Rosen.

> Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

Solution

We will attempt a proof by contradiction. First, assume that both n and $n+2$ are perfect squares. Then, let $n = a^2, n+2 = b^2$ where $a, b \in \mathbb{Z}^+$. From the second equation, subtract 2 from both sides, leaving us with $n = b^2 - 2$. Now, we have two equations where n is isolated, so we can set them equal to each other.

$$a^2 = b^2 - 2$$

Now, add 2 to both sides of the equation.

$$b^2 = 2 + a^2$$

Now, take the square root of both sides.

$$b = \pm\sqrt{2 + a^2}$$

We can reject the negative solution for b, since we restrict to b to the set of all positive integers. This leaves us with the following.

$$b = \sqrt{2 + a^2}$$

Now, inside the radical, factor out a 2 from both terms.

$$b = \sqrt{2 \cdot \left(1 + \frac{a^2}{2}\right)}$$

Now, using exponent rules, we can split the radical into 2 factors.

$$b = \sqrt{2} \cdot \sqrt{1 + \frac{a^2}{2}}$$

$\sqrt{2}$ is an irrational number. An irrational number multiplied by an irrational or rational number will output an irrational number. This means that it does not matter what kind of number $\sqrt{1 + \frac{a^2}{2}}$ is. Therefore, b is irrational. However, this is a contradiction, since we previously specified that $b \in \mathbb{Z}^+$. Thus, our original assumption that $n + 2$ is a perfect square was incorrect. Therefore, by proof of contradiction, if n is a perfect square, then $n + 2$ is not a perfect square. $\square$