Proofs - 1

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November 2, 2023

Prove the following equation using $\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

$$\pi = 4\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{(-1)^n}{2n+1}$$

Part 1: Compute $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

We can solve this integral by recognizing that it represents the area of a sector of a circle. Let $y = \sqrt{9-x^2}$. Now, make the following deductions.

$$y^2 = 9 - x^2 \tag{1}$$
$$x^2 + y^2 = 3^2$$

We now have the equation of a circle, centered at (0,0) and with radius 3. However, we need to know which part of the circle's area to compute. Looking at equation (1), we get the solutions for y, $y = \pm\sqrt{9-x^2}$. Based on the integrand, we can conclude that we need to choose the solution $y = \sqrt{9-x^2}$. This positive solution corresponds to the top half of the circle. The area of the circle is 9π, using the formula $A = \pi r^2$. However, we need to divide this by 2, since we want only the top half. Therefore, $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2}$.

Part 2: Binomial Series Implementation

The integrand of $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ could be written as a binomial series. Begin with the definition of a binomial series.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \;\; \forall\, k \in \mathbb{R}$$

Now, we can make the following deductions.

$$\begin{aligned}
\sqrt{9-x^2} &= \left(9-x^2\right)^{\frac{1}{2}} \\
&= \left(9\left(1-\frac{x^2}{9}\right)\right)^{\frac{1}{2}} \\
&= 3\left(1+\left(-\frac{x^2}{9}\right)\right)^{\frac{1}{2}}, \quad k=\frac{1}{2} \\
&= 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(-\frac{x^2}{9}\right)^n
\end{aligned}$$

$$\sqrt{9-x^2} = 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{x^{2n}}{9^n} \tag{2}$$

Part 3: Completing the Proof

Now, integrate both sides of equation (2). Integrating the infinite series is simple, since we are essentially integrating a polynomial where we can integrate each of the terms using the power rule.

$$\begin{aligned}
\int \sqrt{9-x^2}\,dx &= \int\left[3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{x^{2n}}{9^n}\right]\,dx \\
&= 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{x^{2n+1}}{9^n(2n+1)}
\end{aligned}$$

Now, solve for the definite integral.

$$\begin{aligned}
\int_{-3}^{3}\sqrt{9-x^2}\,dx &= \left[3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{x^{2n+1}}{9^n(2n+1)}\right]\Bigg|_{-3}^{3} \\
&= 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{(3)^{2n+1}}{9^n(2n+1)} - 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{(-3)^{2n+1}}{9^n(2n+1)} \\
&= 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{3^{2n}\cdot 3}{9^n(2n+1)} - 3\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{(-3)^{2n}\cdot(-3)}{9^n(2n+1)} \\
&= 9\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1} + 9\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-1)^n\frac{(-3)^{2n}}{9^n(2n+1)}
\end{aligned}$$

If we examine the $(-3)^{2n}$ component in the second series, notice that the following statement is true.

$$(-3)^{2n} = 3^{2n} \ \ \forall \{n \in \mathbb{Z} | n \geq 0\}$$

Therefore, we can replace $(-3)^{2n}$ with 3^{2n} in the second series.

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9 - x^2}\, dx &= 9 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1} + 9 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{3^{2n}}{9^n(2n+1)} \\
&= 9 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1} + 9 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1} \\
&= 18 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1}
\end{aligned}$$

Recall that the value of the definite integral was $\frac{9\pi}{2}$.

$$\frac{9\pi}{2} = 18 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1}$$

$$\pi = 4 \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-1)^n \frac{1}{2n+1} \quad \square.$$