Integrals - 4

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$$\int \operatorname{sech}(x)\,dx$$

Step 1: Algebraic Manipulation

We can rewrite the integrand by using the following definition.

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{1}{\left(\frac{e^x+e^{-x}}{2}\right)} = \frac{2}{e^x + e^{-x}}$$

Now, continue manipulating the integrand.

$$\begin{aligned}
\int \operatorname{sech}(x)\,dx &= \int \frac{2}{e^x + e^{-x}}\,dx \\
&= \int \frac{2}{e^x + \frac{1}{e^x}}\,dx \\
&= \int \frac{2}{\left(\frac{e^{2x}+1}{e^x}\right)}\,dx \\
&= 2\int \frac{e^x}{e^{2x}+1}\,dx
\end{aligned}$$

Step 2: u-Substitution

We can now continue solving the integral by performing the following u-substitution.

$$u = e^{2x} + 1$$

$$\frac{du}{dx} = 2e^{2x}$$

$$dx = \frac{du}{2e^{2x}}$$

$$2\int \frac{e^x}{u}\,\frac{du}{2e^{2x}}$$

$$\int \frac{1}{ue^x}\,du$$

Since $e^{2x} + 1 = u$, then $e^x = \sqrt{u-1}$. We can now substitute $\sqrt{u-1}$ into the integrand and proceed with the solution.

$$\int \frac{1}{u\sqrt{u-1}}\,du$$

Step 3: Trigonometric Substitution

Now, perform the following trigonometric substitution.

$$u = \sec^2(\theta), \quad du = 2\sec^2(\theta)\tan(\theta)\,d\theta$$

$$\int \frac{1}{u\sqrt{u-1}}\,du = \int \frac{2\sec^2(\theta)\tan(\theta)}{\sec^2(\theta)\sqrt{\sec^2(\theta)-1}}\,d\theta$$

Recall the trigonometric identity: $\sec^2(\theta) - 1 = \tan^2(\theta)$.

$$\int \frac{2\sec^2(\theta)\tan(\theta)}{\sec^2(\theta)\sqrt{\tan^2(\theta)}}\,d\theta$$

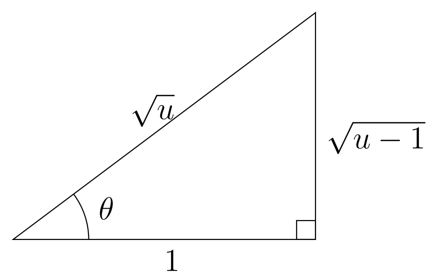
Since we are dealing with the square root of a square in the denominator of the integrand, we should rewrite $\sqrt{\tan^2(\theta)}$ as $|\tan(\theta)|$. We can now set a specific domain for θ, namely $\theta \in (0, \frac{\pi}{2})$, which will allow $|\tan(\theta)|$ to become simply $\tan(\theta)$. Note that we could have also chosen the outcome to be $-\tan(\theta)$, making $\theta \in (-\frac{\pi}{2}, 0)$, but we choose the positive version since it is easier to deal with positive values. Now, we can continue solving the integral.

$$\int \frac{2\sec^2(\theta)\tan(\theta)}{\sec^2(\theta)\tan(\theta)}\, d\theta$$

Now, almost everything cancels nicely, leaving us with the following.

$$\int 2\ d\theta = 2\theta + C$$

We now need to rewrite this result, using the definition we made during the trigonometric substitution.



$$u = \sec^2(\theta), \qquad \sqrt{u} = \sec(\theta)$$

$$\int \frac{1}{u\sqrt{u-1}}\, du = 2\theta + C = 2\,\text{arcsec}\left(\sqrt{u}\right) + C$$

Notice that it seems that we skipped a step and went directly from $u = \sec^2(\theta)$ to $\sqrt{u} = \sec(\theta)$. However, we know that $u = e^{2x} + 1$ and that $u \in (1, \infty)$, which is why we can take the square root of u without considering the negative case resulting from the absolute value.

Finally, substituting in $e^{2x} + 1$ in for the u, we get the answer.

$$\int \text{sech}(x)\ dx = \boxed{2\,\text{arcsec}\left(\sqrt{e^{2x}+1}\right) + C}$$