Integrals - 2

Vignesh Nydhruva

September 14, 2023

$$\int \frac{\sin(x)\cos(x)}{\sin^4(x) + \cos^4(x)}\, dx$$

Step 1: u-Substitution

Although this integral may look a little complicated, it is actually fairly simple. In order to make this integral a little more familiar, we can start off with a u-substitution, which will allow some factors to cancel.

$$u = \sin^2(x)$$

$$\frac{du}{dx} = 2\sin(x)\cos(x)$$

$$dx = \frac{du}{2\sin(x)\cos(x)} = \frac{du}{\sin(2x)}$$

The integrand contains $\sin(x)\cos(x)$ in the numerator and can be rewritten as $\frac{1}{2}\sin(2x)$. Let us now substitute in what we have so far and simplify.

$$\int \frac{\frac{1}{2}\sin(2x)}{u^2 + \cos^4(x)}\, \frac{du}{\sin(2x)}$$

$$\frac{1}{2}\int \frac{1}{u^2 + \cos^4(x)}\, du$$

Now, our differentiable is in terms of u, but there still exists a $\cos^4(x)$ term in the denominator. We can rewrite this in terms of u, using the trigonometric identity $1 - \sin^2(x) = \cos^2(x)$.

$$u = \sin^2(x)$$

$$1 - u = 1 - \sin^2(x) = \cos^2(x)$$

$$(1-u)^2 = \cos^4(x)$$

Now, substitute in $(1-u)^2$, expand, and simplify.

$$\frac{1}{2}\int \frac{1}{u^2 + (1-u)^2}\, du$$

$$\frac{1}{2}\int \frac{1}{u^2 + 1 - 2u + u^2}\, du$$

$$\frac{1}{2}\int \frac{1}{2u^2 - 2u + 1}\, du$$

At this point, it might be tempting to say that we can use partial fractions to solve the integral. However, the denominator of the integrand is not factorable, nor can we use the quadratic formula to find potential real roots.

Step 2: Completing the Square

We can complete the square with the polynomial in the denominator and see where that takes us. First, let us factor out a 2 to get a coefficient of 1 for the u^2 term.

$$\frac{1}{2}\int \frac{1}{2\left(u^2 - u + \frac{1}{2}\right)}\, du$$

$$\frac{1}{4}\int \frac{1}{u^2 - u + \frac{1}{2}}\, du$$

Now, complete the square.

$$\frac{1}{4}\int \frac{1}{\left(u^2 - u + \frac{1}{4}\right) + \frac{1}{4}}\, du$$

$$\frac{1}{4}\int \frac{1}{\left(u - \frac{1}{2}\right)^2 + \frac{1}{4}}\, du$$

Now, factor out $\frac{1}{4}$ in the denominator of the integrand.

$$\frac{1}{4}\int \frac{1}{\frac{1}{4}\left(1+\left(2\left(u-\frac{1}{2}\right)\right)^2\right)}\,du$$

$$\int \frac{1}{1+\left(2\left(u-\frac{1}{2}\right)\right)^2}\,du$$

Step 3: t-Substitution

Now, we can finish the problem by employing a t-substitution.

$$t = 2\left(u-\frac{1}{2}\right) = 2u-1$$

$$\frac{dt}{du} = 2$$

$$du = \frac{dt}{2}$$

Substitute in the respective values and continue solving.

$$\int \frac{1}{1+t^2}\,\frac{dt}{2}$$

$$\frac{1}{2}\int \frac{1}{1+t^2}\,dt$$

$$\frac{1}{2}\arctan(t)+C$$

Finally, substitute in $2u-1$ in for t, and $\sin^2(x)$ in for u.

$$\frac{1}{2}\arctan(2u-1)+C$$

$$\frac{1}{2}\arctan\left(2\left(\sin^2(x)\right)-1\right)+C$$

$$\frac{1}{2}\arctan\left(2\sin^2(x)-1\right)+C$$

$$\frac{1}{2}\arctan\left(-\left(1-2\sin^2(x)\right)\right)+C$$

$$\boxed{\frac{1}{2}\arctan\left(-\cos(2x)\right)+C}$$